

MS455 Assignment 1

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Marks

Each problem is worth 10 marks (with marks divided equally between parts of problems). Marks will be awarded for clear and rigorous answers – clearly cite and justify the use of any auxiliary results that you need.

Submission

Please submit your attempts to the following questions by Friday the 13th of October by 5 pm. Assignments can be submitted directly to me (my office is X138A) or via the assignment submission box outside the School of Mathematical Sciences office.

Problem 1

- (a.) If Ω is a non-empty finite set, then the set of all subsets of Ω , 2^Ω , is also a finite set. Show that 2^Ω is a σ -algebra when Ω is finite.
- (b.) Suppose Ω is an infinite set (countable or uncountable). Consider the collection of sets given by

$$\mathcal{A} = \{A \subset \Omega : A \text{ is finite or } A^c \text{ is finite}\}.$$

Show that \mathcal{A} is **not** a σ -algebra.

HINT: Consider what could go wrong when taking countable unions of events in \mathcal{A} .

Problem 2

- (a.) Let $X \sim \mathcal{N}(0, 1)$ and derive the density of the random variable X^2 .
- (b.) Let X_1, \dots, X_n be $\mathcal{N}(0, 1)$ random variables which are mutually independent and derive the density of $\sum_{i=1}^n X_i^2$ for any $n \in \mathbb{N}$.
- HINT: You could proceed by induction.

Problem 3

For each $n \in \mathbb{N}$, let X_n be a random variable with p.d.f given by

$$f_n(x) = \frac{n}{\pi(1+n^2x^2)}, \quad x \in \mathbb{R}.$$

- (a.) Show that the sequence $\{X_n\}_{n \geq 1}$ does **not** tend to zero in L^2 .
- (b.) Calculate the c.d.f of X_n for each $n \in \mathbb{N}$ and hence show that $\{X_n\}_{n \geq 1}$ tends to zero in probability.

Problem 4

Let $n \in \mathbb{N}$ and X_1, \dots, X_n be mutually independent Gamma distributed random variables with parameters $\alpha > 0$ and $\beta > 0$. Note that the p.d.f. of a Gamma random variable is given by

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

For each $n \in \mathbb{N}$, define the random variable A_n by

$$A_n = B \left(\sum_{i=1}^n X_i \right) + C,$$

where B and C are positive constants.

- (a.) Calculate the mean μ and variance σ^2 of A_n .
- (b.) Calculate a 95% asymptotic confidence interval for A_n and clearly explain the theoretical justification for your answer.