MS555 Assignment 1

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25th September, 2017

Marks

Each problem is worth 8 marks (with marks divided equally between parts of problems). Marks will be awarded for clear and rigorous answers – clearly cite and justify the use of any auxiliary results that you need.

Submission

Please submit your attempts to the following questions by Friday the 13th of October by 5 pm. Assignments can be submitted directly to me (my office is X138A) or via the assignment submission box outside the School of Mathematical Sciences office.

Problem 1

(a.) If Ω is a non–empty finite set, then the set of all subsets of Ω, 2^Ω, is also a finite set. Show that 2^Ω is a σ–algebra when Ω is finite.

(b.) Suppose Ω is an infinite set (countable or uncountable). Consider the collection of sets given by

$$\mathcal{A} = \{A \subset \Omega : A \text{ is finite or } A^c \text{ is finite}\}.$$

Show that $\mathcal{A}$ is **not** a σ–algebra.
HINT: Consider what could go wrong when taking countable unions of events in $\mathcal{A}$.

Problem 2

(a.) Let $X \sim \mathcal{N}(0, 1)$ and derive the density of the random variable X^2.

(b.) Let $X_1, \dots, X_n$ be $\mathcal{N}(0, 1)$ random variables which are mutually independent and derive the density of $\sum_{i=1}^{n} X_i^2$ for any $n \in \mathbb{N}$.
HINT: You could proceed by induction.

Problem 3

For each $n \in \mathbb{N}$, let X_n be a random variable with p.d.f given by

$$f_n(x) = \frac{n}{\pi\,(1 + n^2 x^2)}, \quad x \in \mathbb{R}.$$

(a.) Show that the sequence $\{X_n\}_{n\geq 1}$ does **not** tend to zero in L^2.

(b.) Calculate the c.d.f of X_n for each $n \in \mathbb{N}$ and hence show that $\{X_n\}_{n\geq 1}$ tends to zero in probability.

Problem 4

Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be mutually independent Gamma distributed random variables with parameters $\alpha > 0$ and $\beta > 0$. Note that the p.d.f. of a Gamma random variable is given by

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

For each $n \in \mathbb{N}$, define the random variable A_n by

$$A_n = B\left(\sum_{i=1}^{n} X_i\right) + C,$$

where B and C are positive constants.

(a.) Calculate the mean μ and variance σ^2 of A_n.

(b.) Calculate a 95% asymptotic confidence interval for A_n and clearly explain the theoretical justification for your answer.

Problem 5

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d $\mathcal{N}(1,3)$ random variables. Use the Strong Law of Large Numbers to show that

$$\frac{X_1 + X_2 + \ldots + X_n}{X_1^2 + X_2^2 + \ldots + X_n^2}$$

converges almost surely to a nonzero constant as $n \to \infty$.