MS555 Assignment 1 Solutions

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Problem 1

Definition 1 (σ–algebra)**.** If Ω is any non–empty set, then Σ is a σ–algebra on Ω if it has the following properties:

(i.) $\emptyset \in \Sigma$ and $\Omega \in \Sigma$,

(ii.) if $E \in \Sigma$, then $E^c \in \Sigma$ (where E^c is the complement of E),

(iii.) if $\{E_n\}_{n\geq 1}$ are all in Σ, then $\bigcup_{n=1}^{\infty} E_n \in \Sigma$.

(a.) (i.) Since $\emptyset \subset \Omega$ and $\Omega \subseteq \Omega$, we immediately have that $\emptyset \in 2^\Omega$ and $\Omega \in 2^\Omega$.

(ii.) If $E \in 2^\Omega$, is $E^c \in 2^\Omega$? Since A^c is also a subset of Ω it must be in 2^Ω also.

(iii.) Note that we have not yet used the key hypothesis in this problem, i.e. that Ω is finite! Suppose $\{E_n\}_{n\geq 1} \in 2^\Omega$. Since 2^Ω is finite,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{K\leq 2^N} E_{\alpha(n)},$$

where $N = |\Omega|$ and $\alpha : \mathbb{N} \mapsto \mathbb{N}$ tracks any rearrangement of indices. In other words, the countable union above must contain repeated sets (infinitely many) since there are only finitely many sets in 2^Ω. It follows easily that 2^Ω is closed under unions and hence finite unions as well.

(b.) (i.) $\Omega^c = \emptyset$ is finite so $\emptyset \in \mathcal{A}$ and $\Omega \in \mathcal{A}$.

(ii.) If $A \in \mathcal{A}$, then one of the following is true:

- A finite, A^c infinite;
- A infinite, A^c finite.

In each case, we have $A^c \in \mathcal{A}$ also.

(iii.) This where it all goes wrong for $\mathcal{A}$. First note that

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A} \iff \bigcup_{n=1}^{\infty} A_n \quad \text{is finite} \quad \textbf{or} \quad \bigcap_{n=1}^{\infty} A_n^c \quad \text{is finite.}$$

The key point is that a **finite** union of finite sets is finite but a **countable** union of finite sets may be infinite; the easiest way to see this is to take a concrete example. Take $\Omega = \mathbb{R}$ and consider the sequence of sets $A_n = \{n\}$ for each $n \in \mathbb{N}$. Each A_n is a finite set (a point) so $\{A_n\}_{n\geq 1} \in \mathcal{A}$. However,

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{N},$$

i.e. an infinite set. Furthermore, $A_n^c = \mathbb{R}/\{n\}$ for each $n \in \mathbb{N}$. Hence

$$\bigcap_{n=1}^{\infty} A_n^c = \bigcap_{n=1}^{\infty} \mathbb{R}/\{n\} = \mathbb{R}/\mathbb{N},$$

which is still an infinite set and thus $\bigcup\limits_{n=1}^{\infty} A_n \notin \mathcal{A}$.

Problem 2

(a.) The p.d.f. of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Thus the c.d.f. of X^2 is

$$F_{X^2}(x) = \mathbb{P}\left[X^2 \leq x\right] = 0, \quad x \leq 0,$$

since X^2 cannot take on negative values. Furthermore,

$$F_{X^2}(x) = \mathbb{P}\left[X^2 \leq x\right] = \mathbb{P}\left[-\sqrt{x} \leq X \leq \sqrt{x}\right] = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du, \quad x > 0.$$

Now differentiate the expression above to obtain the p.d.f. for X^2:

$$\frac{d}{dx} \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2} =: f_{X^2}(x), \quad x > 0.$$

Therefore

$$f_{X^2}(x) = \begin{cases} \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

(b.) Miraculously, we manage to guess that the p.d.f. of $Y_k := \sum_{i=1}^{k} X_i^2$ is given by

$$f_{Y_k}(x) = \begin{cases} \frac{x^{(k/2)-1}}{2^{k/2}\Gamma(k/2)} e^{-x/2}, & x > 0, \\ 0, & x \leq 0, \end{cases} \qquad (H_k)$$

where $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$. In part (a.), we showed that (H_k) is true for $k = 1$. Now suppose H_k holds for some $k \geq 1$ and try to show that (H_{k+1}) is true as a consequence. The c.d.f. of Y_{k+1} can be computed as follows:

$$\begin{aligned}\mathbb{P}[Y_{k+1} \leq x] &= \mathbb{P}[Y_k \leq x - X_{k+1}^2] = \int_0^x \mathbb{P}[Y_k \leq x - s \,|\, X_{k+1}^2 = s] f_{X_{k+1}^2}(s)\,ds \\ &= \int_0^x \int_0^{x-s} f_{Y_k}(u)\,du\, f_{X_{k+1}^2}(s)\,ds \quad \text{(by independence)}, \quad x > 0.\end{aligned}$$

Hence

$$f_{Y_{k+1}}(x) = \frac{d}{dx}\mathbb{P}[Y_{k+1} \leq x] = \int_0^x f_{Y_k}(x-s) f_{X_{k+1}^2}(s)\,ds = \int_0^x f_{Y_k}(s) f_{X_{k+1}^2}(x-s)\,ds,$$

by Leibnitz rule. Now we just need to resolve the final integral on the right–hand side above under (H_k); the change of variable $u = s/x$ is crucial. Calculate as follows:

$$\begin{aligned}f_{Y_{k+1}}(x) &= \frac{1}{2^{k/2}\Gamma(k/2)\sqrt{2\pi}} \int_0^x s^{k/2-1}e^{-s/2}(x-s)^{-1/2}e^{-(x-s)/2}\,ds \\ &= \frac{e^{-x/2}}{2^{k/2}\Gamma(k/2)\sqrt{2\pi}} \int_0^x s^{k/2-1}(x-s)^{-1/2}\,ds \\ &= \frac{e^{-x/2}}{2^{k/2}\Gamma(k/2)\sqrt{2\pi}} \int_0^1 (ux)^{k/2-1}x^{1/2}(1-u)^{-1/2}\,du \\ &= \frac{e^{-x/2}x^{(k+1)/2-1}}{2^{k/2}\Gamma(k/2)\sqrt{2\pi}} \int_0^1 u^{k/2-1}(1-u)^{3/2-1}\,du \\ &= \frac{e^{-x/2}x^{(k+1)/2-1}}{2^{(k+1)/2}\Gamma(k/2)\sqrt{\pi}} \frac{\Gamma(k/2)\Gamma(1/2)}{\Gamma((k+1)/2)} \\ &= \frac{e^{-x/2}x^{(k+1)/2-1}}{2^{(k+1)/2}\Gamma((k+1)/2)},\end{aligned}$$

as required. Note that $\Gamma(1/2) = \sqrt{\pi}$.

Problem 3

(a.) Recall that $X_n \to 0$ in L^2 if and only if $\lim_{n\to\infty} \mathbb{E}[X_n^2] = 0$. However, in this case

$$\mathbb{E}[X_n^2] = \frac{n}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+n^2x^2} = \infty, \quad n \in \mathbb{N},$$

because $\lim_{|x|\to\infty} x^2/(1+n^2x^2) = 1$, i.e. the integrand is positive and doesn't tend to zero at $\pm\infty$ so the integral diverges. Hence $X_n \not\to 0$ in L^2.

(b.) Firstly, calculate the c.d.f. of each X_n as follows:

$$\begin{aligned}\mathbb{P}[X_n \leq x] &= \frac{n}{\pi}\int_{-\infty}^{x} \frac{du}{1+n^2u^2} = \frac{1}{\pi}\int_{-\infty}^{n} x\frac{dy}{1+y^2} = \frac{\arctan(nx)}{\pi} - \frac{\arctan(-\infty)}{\pi} \\ &= \frac{\arctan(nx)}{\pi} + \frac{1}{2}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.\end{aligned}$$

Now recall that $X_n \to 0$ in probability if and only if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[|X_n| < \epsilon] = 1.$$

Let $\epsilon > 0$ be arbitrary and calculate as follows:

$$\mathbb{P}[|X_n| < \epsilon] = \mathbb{P}[-\epsilon < X_n < \epsilon] = \mathbb{P}[X_n \leq \epsilon] - \mathbb{P}[X_n \leq -\epsilon] = \frac{\arctan(n\epsilon)}{\pi} - \frac{\arctan(-n\epsilon)}{\pi}.$$

Note that $\lim_{x\to\infty} \arctan(x) = \pi/2$ and $\lim_{x\to-\infty} \arctan(x) = -\pi/2$. Therefore

$$\lim_{n\to\infty} \mathbb{P}[|X_n| < \epsilon] = \lim_{n\to\infty} \left(\frac{\arctan(n\epsilon)}{\pi} - \frac{\arctan(-n\epsilon)}{\pi} \right) = 1,$$

i.e. $X_n \to 0$ in probability.

Problem 4

(a.) First note that since the X_i's are Gamma distributed we have

$$\mathbb{E}[X_i] = \frac{\alpha}{\beta}, \quad \mathrm{Var}[X_i] = \frac{\alpha}{\beta^2}, \quad i \in \mathbb{N}.$$

The mean and variance of A_n can then be computed directly as follows:

$$\mathbb{E}[A_n] = B \sum_{i=1}^{n} \mathbb{E}[A_n] + C = n\, B\, \frac{\alpha}{\beta} + C =: \mu.$$

$$\mathrm{Var}[A_n] = B^2 \sum_{i=1}^{n} \mathrm{Var}[A_n] = n\, B^2\, \frac{\alpha}{\beta^2} =: \sigma^2.$$

(b.) Since A_n is a sum of i.d.d. random variables with **finite mean and variance** the Central Limit theorem tells us that the A_n's converge in distribution to a normal random variable with mean μ and variance σ^2. Hence an 95% asymptotic confidence interval is given by

$$(\mu - 1.96\sigma,\, \mu + 1.96\sigma) = \left(n\, B\, \frac{\alpha}{\beta} + C - \frac{1.96B}{\beta}\sqrt{n\alpha},\, n\, B\, \frac{\alpha}{\beta} + C + \frac{1.96B}{\beta}\sqrt{n\alpha} \right).$$

Problem 5

From the question statement we know that $\mathbb{E}[X_i] = 1$ and $\mathrm{Var}[X_i] = 3$. Furthermore,

$$\mathrm{Var}[X_i] = 3 = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = \mathbb{E}[X_i^2] - 1.$$

Hence $\mathbb{E}[X_i^2] = 4$. You can also show from here that $\mathrm{Var}[X_i^2] < \infty$ with little trouble. Now apply the Strong Law of Large Numbers as follows:

$$\frac{X_1 + \ldots X_n}{X_1^2 + \ldots X_n^2} = \frac{X_1 + \ldots X_n}{n} \times \frac{n}{X_1^2 + \ldots X_n^2} \longrightarrow 1 \times \frac{1}{4} = \frac{1}{4} \quad a.s. \text{ as } n \to \infty.$$