Math 20A: Multivariable Calculus

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Chapter 1

Vectors and Geometry

Definition 1 *The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, i.e.*

$$A \times B := \{(a, b) : a \in A,\, b \in B\}.$$

Definition 2 $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x \in \mathbb{R},\, y \in \mathbb{R},\, z \in \mathbb{R}\}$.

Definition 3 *A surface S in $\mathbb{R}^3$ is a set of points obeying a relation of the form $f(x, y, z) = 0$ for some smooth function $f : \mathbb{R}^3 \mapsto \mathbb{R}$.*

Example 1 (Surfaces in $\mathbb{R}^3$) *A sphere of radius r with center $(x_0,\, y_0,\, z_0)$ is the set of all points (x, y, z) satisfying the equation $(x - x_0)^2 + (y - y_0)^2 + (z - z_0^2) = r^2$.*

A cylinder in $\mathbb{R}^3$ centered on the z–axis with radius r and height h is the set of all points (x, y, z) obeying $x^2 + y^2 = r^2$ and $0 \leq z \leq h$. The base of this cylinder sits on the xy–plane.

Definition 4 *The Euclidean distance between two points, say $(x_0, z, y_0,\, z_0)$ and $(x_1,\, y_1,\, z_1)$, in $\mathbb{R}^3$ is given by the formula*

$$d\left((x_0, y_0, z_0), (x_1, y_1, z_1)\right) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

Definition 5 *A vector is a quantity with both a magnitude and a direction (e.g. a force). We will consider points in $\mathbb{R}^3$ as vectors – the direction of a vector $\vec{a} = (x, y, z)$ is the direction of the line segment connecting $(0, 0, 0)$ to (x, y, z) and the magnitude of the vector $\vec{a}$ is the length of this line segment, i.e. the magnitude of $\vec{a}$ is given by*

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}.$$

Definition 6 (Vector operations) *If $\vec{a} = (a_1,\, a_2,\, a_3)$ and $\vec{b} = (b_1,\, b_2,\, b_3)$ are vectors in $\mathbb{R}^3$, then*

$$\vec{a} + \vec{b} = (a_1 + b_1,\, a_2 + b_2,\, a_3 + b_3) \qquad \textit{[Vector addition]}$$

and

$$c\,\vec{a} = (c\,a_1,\, c\,a_2,\, c\,a_3) \quad \textit{for each } c \in \mathbb{R}, \qquad \textit{[Scalar multiplication]}$$

where the number c is referred to as a scalar in this context.

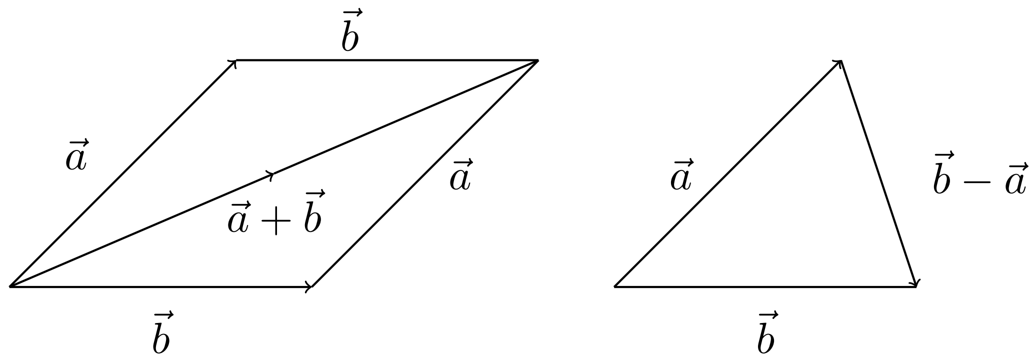

Figure 1.1: Left: Parallelogram law for vector addition. Right: $\vec{b}-\vec{a}$ is the vector corresponding the line segment connecting the endpoint of $\vec{a}$ to the endpoint of $\vec{b}$.

Proposition 1 (Properties of Vector Operations) *If $\vec{a}$ and $\vec{b}$ denote vectors and $\vec{0}=(0,0,0)$ denotes the zero vector, then*

(i) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) $\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}$

(iii) $\vec{a}+\vec{0}=\vec{a}$

(iv) $\vec{a}+(-\vec{a})=\vec{0}$

(v) $c\left(\vec{a}+\vec{b}\right)=c\vec{a}+c\vec{b}$

(vi) $(c+d)\,\vec{a}=c\vec{a}+d\vec{a}$

(vii) $(cd)\vec{a}=c(d\vec{a})$

(viii) $1\times\vec{a}=\vec{a}$

Definition 7 *The standard basis vectors in $\mathbb{R}^3$ are given by*

$$\vec{i}=(1,\,0,\,0),\quad \vec{j}=(0,\,1,\,0),\quad \vec{k}=(0,\,0,\,1).$$

Any vector $\vec{a}=(a_1,\,a_2,\,a_3)$ can be represented in terms of the standard basis vectors as follows:

$$\vec{a}=a_1\,\vec{i}+a_2\,\vec{j}+a_3\,\vec{k}$$

Definition 8 (Dot Product) *The dot product of two vectors $\vec{a}=(a_1,\,a_2,\,a_3)$ and $\vec{b}=(b_1,\,b_2,\,b_3)$ is the scalar $a\cdot\vec{b}$ whose value is given by the formula*

$$\vec{a}\cdot\vec{b}=a_1\,b_1+a_2\,b_2+a_3\,b_3.$$

Proposition 2 (Properties of the Dot Product) *Suppose that veca and vecb are vectors and $c\in\mathbb{R}$. Then*

1. $\vec{a}\cdot\vec{a}=|a|^2$

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3. $\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) \cdot \vec{c}$

4. $(c\,\vec{a}) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right) = \vec{a} \cdot \left(c\,\vec{b}\right)$

5. $\vec{0} \cdot \vec{a} = 0$

Theorem 1 *If $\vec{a}$ and $\vec{b}$ are vectors with an angle θ between them, then*

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta,$$

and hence, if $\vec{a}$ and $\vec{b}$ are both nonzero,

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

Corollary 1 *Two vectors are said to be orthogonal if the angle between them is 90 degrees (the zero vector is considered orthogonal to every other vector).*

Two vectors are orthogonal if and only if their dot product is zero.

Definition 9 *Let $\vec{a}$ and $\vec{b}$ denote vectors. The scalar projection of $\vec{b}$ onto $\vec{a}$ is given by*

$$comp_{\vec{a}}\,\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \qquad [scalar]$$

and the vector projection of $\vec{b}$ onto $\vec{a}$ is given by

$$proj_{\vec{a}}\,\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\vec{a} \qquad [vector].$$

Definition 10 *The cross product of the vectors $\vec{a} = (a_1,\, a_2,\, a_3)$ and $\vec{b} = (b_1,\, b_2,\, b_3)$ is the vector $\vec{a} \times \vec{b}$ given by the formula*

$$\vec{a} \times \vec{b} = (a_2\, b_3 - b_2\, a_3,\; b_1\, a_3 - a_1\, b_3,\; a_1\, b_2 - a_2\, b_1)\,.$$

$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$ and its direction is given by the "right–hand rule".

Theorem 2 *If $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$ with an angle θ between them, then*

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin\theta.$$

Corollary 2 *Two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ are parallel if and only if $\vec{a} \times \vec{b} = 0$.*

Proposition 3 (Properties of the Cross Product) *Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ denote vectors in $\mathbb{R}^3$, and c a scalar. Then*

(i.) $\vec{a} \times \vec{b} = -\,\vec{b} \times a$

(ii.) $(c\,\vec{a}) \times \vec{b} = c\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(c\,\vec{b}\right)$

(iii.) $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(iv.) $\left(\vec{a} + \vec{b}\right) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

(v.) $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$

(vi.) $\vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$

Proposition 4 *In $\mathbb{R}^3$, the volume V of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is the magnitude of their scalar triple product, which is given by*

$$V = \left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$$

Chapter 2

Vector–valued Functions

Definition 11 *The parametric equation of the line L through the point $\vec{r}_0 = (x_0,\, y_0,\, z_0)$ with direction vector $\vec{v} = (a,\, b,\, c)$ is given by*

$$\vec{r}(t) = (x_0 + at,\, y_0 + bt,\, z_0 + ct) = \vec{r}_0 + t\,\vec{v}, \quad t \in \mathbb{R}.$$

The symmetric equations of the line L are given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

provided that a, b and c are all nonzero.

Definition 12 *The plane through $\vec{r}_0 = (x_0,\, y_0,\, z_0)$ with normal vector $\vec{n} = (a,\, b,\, c)$ is given by the set of vectors $\vec{r} = (x,\, y,\, z)$ obeying the equation*

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Proposition 5 *The distance from the point $(x_1,\, y_1,\, z_1)$ to the plane with equation $ax + by + cz = d$ is given by*

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Definition 13 *A vector–valued function $\vec{r} : \mathbb{R} \mapsto \mathbb{R}^3$ is given by*

$$\vec{r}(t) = (x(t),\, y(t),\, z(t))\,, \quad t \in \mathbb{R},$$

where $x : \mathbb{R} \mapsto \mathbb{R}$, $y : \mathbb{R} \mapsto \mathbb{R}$ and $z : \mathbb{R} \mapsto \mathbb{R}$ are called the component functions of $\vec{r}$.

Definition 14 (Limits and Continuity) *If $\vec{r}(t) = (x(t),\, y(t),\, z(t))$, then*

$$\lim_{t\to t_0} \vec{r}(t) = \left(\lim_{t\to t_0} x(t),\, \lim_{t\to t_0} y(t),\, \lim_{t\to t_0} z(t)\right),$$

assuming the componentwise limits exist and $\vec{r}$ is continuous at $a \in \mathbb{R}$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Definition 15 *If $\vec{r}(t) = (x(t),\, y(t),\, z(t))$, then the derivative of $\vec{r}$ is given by*

$$\frac{d}{dt}\vec{r}(t) = \left(x'(t),\, y'(t),\, z'(t)\right),$$

assuming the component functions of $\vec{r}$ are differentiable.

Proposition 6 (Derivatives of Vector Functions) *Let $\vec{u}$ and $\vec{v}$ be vector valued functions, f a scalar function and c a scalar. Then*

(i) $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$

(ii) $\frac{d}{dt}\left(c\, u\vec{(t)}\right) = c\frac{d}{dt}\vec{u}(t)$

(iii) $\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\frac{d}{dt}\vec{u}(t)$

(iv) $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\frac{d}{dt}\vec{v}(t)$

(v) $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t)\times\vec{v}(t) + \vec{u}(t)\times\frac{d}{dt}\vec{v}(t)$

(vi) $\frac{d}{dt}\left(\vec{u}\left(f(t)\right)\right) = f'(t)\vec{u}'\left(f(t)\right)$

Suppose that $\vec{r}(t) = (x(t),\, y(t),\, z(t))$, where $x : \mathbb{R} \mapsto \mathbb{R}$, $y : \mathbb{R} \mapsto \mathbb{R}$ and $z : \mathbb{R} \mapsto \mathbb{R}$. We also ask that x', y' and z' are continuous.

Proposition 7 (Integration of vector functions)

$$\int_a^b \vec{r}(t)\, dt = \left(\int_a^b x(t)\, dt,\, \int_a^b y(t)\, dt,\, \int_a^b z(t)\, dt\right), \quad a, b \in \mathbb{R}.$$

Definition 16 *The length of the curve $C = \{(x, y, z) : (x, y, z) = \vec{r}(t)$ for some $t \in [a,\, b]\}$ is given by*

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + + (z'(t))^2}\, dt = \int_a^b \left|\vec{r}'(t)\right|\, dt.$$

Definition 17 *The arc length function s measures the length of the curve between a starting point $t = a$ and some variable endpoint:*

$$s(t) := \int_a^t \left|\vec{r}'(t)\right|\, dt, \quad t \in [a,\, b].$$

A curve traced out by $\vec{r}$ is parameterised with respect to arc length by computing the new parameterisation $\vec{\rho}$:

$$\vec{\rho}(t) = \vec{r}(s^{-1}(t)).$$

Definition 18 (Curvature) *The curve C traced out by $\vec{r}$ is smooth if x', y' and z' are continuous and $\vec{r}'(t) \neq 0$. The unit tangent vector to C at a point t is then given by*

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

The curvature of C at t is denoted by κ and is given by

$$\kappa(t) := \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\vec{T}\,'(t)\right|}{|\vec{r}\,'(t)|},$$

where s denotes the arc length function.

Theorem 3 *The curvature can be expressed directly in terms of the vector fuction $\vec{r}$ which paramterises the curve as follows (if $\vec{r}$ is twice differentiable):*

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}.$$

Definition 19 *The normal vector $\vec{N}$ and Binormal vector $\vec{B}$ to a curve traced out by a function $\vec{r}$ are given by*

$$\vec{N}(t) := \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|}, \quad \vec{B}(t) := \vec{T}(t) \times \vec{N}(t).$$

The normal plane *to a curve at a point is the plane containing the vectors $\vec{N}$ and $\vec{B}$ – it contains all lines orthogonal to $\vec{T}$. The* osculating plane *to a curve at a given point is the plane which contains $\vec{T}$ and $\vec{N}$ – it is the plane that comes closest to containing the part of the curve at the chosen point.*

Chapter 3

Partial Differentiation

Definition 20 *For a given natural number n we define the set* $\mathbb{R}^n$ *to be the* n*–fold Cartesian product of* $\mathbb{R}$ *with itself, i.e*

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \{(x_1, x_2, \ldots, x_n) : x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}.$$

Definition 21 *A map* $f : D \subset \mathbb{R}^n \mapsto \mathbb{R}$ *is called a function of* n *variables.* f *is a rule which assigns to each* n*–tuple of numbers* $(x_1, \ldots, x_n) \in D \subset \mathbb{R}^n$*, a number* $f(x_1, \ldots, x_n) \in \mathbb{R}$*. The set* $D \subset \mathbb{R}^n$ *is the domain of* f *and the set* $\{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in D\}$ *is the range of* f*.*

Definition 22 *The graph of a function* $f : D \subset \mathbb{R}^n \mapsto \mathbb{R}$ *is the set*

$$\{(x_1, \ldots, x_{n+1}) : (x_1, \ldots, x_n) \in D,\, x_{n+1} = f(x_1, \ldots, x_n)\}.$$

Definition 23 (Level Curves) *The level curves of a function* $f : D \subset \mathbb{R}^n \mapsto \mathbb{R}$ *are the curves defined by equations of the form*

$$k = f(x_1, \ldots, x_n)$$

for each $k \in \mathbb{R}$*.*

Definition 24 (Limits) *Let* f *be a function of two variables with domain* $D \in \mathbb{R}^2$ *such that* $(a, b) \in D$*. We say the limit of* f *as* (x, y) *tends to* (a, b) *is* $L \in \mathbb{R}$ *and we write*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|(x, y) - (a, b)| < \delta \quad \Rightarrow \quad |f(x, y) - L| < \epsilon.$$

Definition 25 (Continuity) *Let* f *be a function of two variables with domain* $D \in \mathbb{R}^2$ *such that* $(a, b) \in D$*. We say that* f *is continuous at* (a, b) *if*

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Definition 26 (Partial Differentiation) *Let $f(x,y)$ be a function of two variables with domain $D \in \mathbb{R}^2$ such that $(a,b) \in D$. The partial derivative of f with respect to x at (a,b) is dentoed by $f_x(a,b)$ or $\partial_x f(a,b)$ and is given by*

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

The partial derivative of f with respect to y at (a,b) is dentoed by $f_y(a,b)$ or $\partial_y f(a,b)$ and is given by

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Theorem 4 (Clairaut–Schwarz Theorem) *Suppose f is defined on a disk $D \subset \mathbb{R}^2$ containing (a,b). If f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Definition 27 (Tangent Plane) *Let $f(x,y)$ be a function of two variables with domain $D \subset \mathbb{R}^2$ with continuous partial derivatives. The equation of the tangent plane to the surface defined by $z = f(x,y)$ at (x_0, y_0, z_0) is given by*

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

Definition 28 (Differentiability) *A function $f : (x,y) \mapsto f(x,y)$ is differentiable at a point (a,b) if*

$$f(a+\Delta x, b+\Delta y) - f(a,b) = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem 5 *If the partial derivatives of f exist and are continuous in a neighborhood of a point, then f is differentiable at that point.*

Theorem 6 (Chain Rule – two variables) *Suppose that $z = f(x,y)$ is differentiable in both x and y. If $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then*

$$\frac{dz}{dt} = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Theorem 7 (Chain Rule – n variables) *Suppose that u is a differentiable function of n variables $x_1, \ldots, x_n$ and that each variable x_j is a differentiable function of m variables $t_1, \ldots, t_m$. Then*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i},$$

for each $i \in \{1, \ldots, m\}$.

Definition 29 (Directional Derivative) *The directional derivative of a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ at a point (x_0, y_0) in the direction $\vec{u} = (a,b)$ is given by*

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0,y_0)}{h},$$

assuming that the limit exists.

Theorem 8 *The directional derivative of a function* $f : \mathbb{R}^2 \mapsto \mathbb{R}$ *at a point* (x_0, y_0) *in the direction* $\vec{u} = (a, b)$ *can be expressed as*

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Definition 30 (Gradient) *If* $f : \mathbb{R}^n \mapsto \mathbb{R}$, *i.e.* $f : (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$, *is differentiable with respect to each coordinate, then the gradient of* f *at the point* $(a_1, a_2, \ldots, a_n)$ *is given by*

$$\nabla f(a_1, a_2, \ldots, a_n) = \left(f_{x_1}(a_1, a_2, \ldots, a_n), \ldots, f_{x_n}(a_1, a_2, \ldots, a_n)\right).$$

In two dimensions, the gradient of $f(x, y)$ *at* (x_0, y_0) *is* $\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$.

Theorem 9 *The directional derivative* $D_{\vec{u}} f$ *achieves it's maximum when* $\vec{u}$ *has the same direction as* ∇f *and the maximum value of the directional derivative is* $|\nabla f|$.

Let $f : (x, y) \mapsto f(x, y)$ be a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$.

Definition 31 f *has a local maximum at* (a, b) *if there exists an* $\epsilon > 0$ *such that* $f(x, y) \leq f(a, b)$ *for all* (x, y) *obeying* $\sqrt{(x - a)^2 - (y - b)^2} < \epsilon$. *More usually we will say that* $f(x, y) \leq f(a, b)$ *for all* (x, y) *in some neighborhood of* (a, b) *– the neighborhood being referred to is the disk of radius* ϵ *centered at* (a, b).

f *has a local minimum at* (a, b) *if there exists an* $\epsilon > 0$ *such that* $f(x, y) \geq f(a, b)$ *for all* (x, y) *obeying* $\sqrt{(x - a)^2 - (y - b)^2} < \epsilon$.

Definition 32 f *has a critical point at* (a, b) *if* $f_x(a, b) = f_y(a, b) = 0$.

Theorem 10 *Local maxima and minima occur at critical points. In other words, if* (a, b) *is a local maximum or minimum for* f, *then it must be the case that*

$$f_x(a, b) = f_y(a, b) = 0.$$

Definition 33 *If the second order partial derivatives of* f *exist and are continuous in a neighborhood of* (a, b), *then the Hessian matrix of* f *at* (a, b) *is given by the* 2×2 *matrix*

$$H_f(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix}$$

Theorem 11 *Suppose* f *has a critical point at* (a, b) *and that the second order partial derivatives of* f *exist and are continuous in a neighborhood of* (a, b). *Define*

$$D = D(a, b) = \det\left[H_f(a, b)\right] = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2.$$

(i.) *If* $D > 0$ *and* $f_{xx}(a, b) > 0$, *then* (a, b) *is a local minimum of* f.

(ii.) *If* $D > 0$ *and* $f_{xx}(a, b) < 0$, *then* (a, b) *is a local maximum of* f.

(iii.) *If* $D < 0$, *then* (a, b) *is neither a local minimum or a local maximum but is called a saddle point of* f.

Theorem 12 *If Ω is a closed and bounded set in $\mathbb{R}^2$ and f is continuous, then f attains it's maximum and minimum values on Ω.*

Maximizing/minimizing continuous functions on closed, bounded domains:

1. Find the values of the function at the critical points,
2. Find the extreme values of the function on the boundaries of the domain,
3. Find the largest/smallest of the functions values in the previous steps.

Definition 34 *A Lagrange multiplier problem is a maximization (resp. minimization) problem of the following form:*

$$\max_{(x,y)\in A} f(x,y), \quad A = \{(x,y) : g(x,y) = k \in \mathbb{R}\}.$$

Method of Lagrange Multipliers: Let f and g be differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\nabla g \neq 0$ on the surface $g(x,y) = k$. Then in order to solve the Lagrange multiplier problem for f and g:

(i.) Find all the values of x, y and λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = k,$$

(ii.) Evaluate f at the points found in step (i.) — the largest value of f is the maximum, the smallest is the minimum.

The number λ is called the Lagrange multiplier.

Chapter 4

Multiple Integrals

Definition 35 *If $D \subset \mathbb{R}^2$, we denote the area of D by $A(D)$. We denote the boundary of D by ∂D – recall that a point is a boundary point of D if every neighborhood of the point contains points both inside and outside of D.*

Definition 36 *Let $f : R^2 \mapsto \mathbb{R}$ be a continuous function, where*

$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}.$$

Consider a partition of $[a, b]$ of the form $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ and a partition of $[c, d]$ of the form $c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$. Choose these partitions such that $x_i - x_{i-1} = (b-a)/m =: \Delta x$ for each $i = 1, \ldots, m$ and $y_j - y_{j-1} = (d-c)/n =: \Delta y$ for each $j = 1, \ldots, n$ so that each rectangle $R_{i,j} = [x_i, x_{i-1}] \times [y_j, y_{j-1}]$ has area $\Delta A = \Delta x \Delta y$. We define the integral of f over R by the following limit

$$\iint_R f(x, y)\, dA := \lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta A$$

where x_i^* denotes a point in the interval $[x_{i-1}, x_i]$ and y_j^* denotes a point in the interval $[y_{j-1}, y_j]$.

Theorem 13 (Fubini) *Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a continuous function and let*

$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}.$$

Then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Definition 37 *$D \subset \mathbb{R}^2$ is called a type–1 region if it can be expressed in the form*

$$D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

for continuous scalar functions g_1 and g_2.

$E \subset \mathbb{R}^2$ is called a type–2 region if it can be expressed in the form

$$E = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\},$$

for continuous scalar functions h_1 and h_2.

An elementary region is a region which is either type–1, type–2 or rectangular.

Proposition 8 *Let $f, g : \mathbb{R}^2 \mapsto \mathbb{R}$ be continuous functions and suppose $D \subset \mathbb{R}^2$ is an elementary region.*

(i.) $\iint_D f(x, y) + g(x, y)\, dx\, dy = \iint_D f(x, y)\, dx\, dy + \iint_D g(x, y)\, dx\, dy.$

(ii.) $\iint_D c\, f(x, y)\, dx\, dy = c \iint_D f(x, y)\, dx\, dy,$ *where* $c \in \mathbb{R}$.

(iii.) If $g(x, y) \leq f(x, y)$ for each $(x, y) \in D$, then $\iint_D g(x, y)\, dx\, dy \leq \iint_D f(x, y)\, dx\, dy.$

(iv.) If D_1 and D_2 are elementary regions such that $D = D_1 \cup D_2$, then

$$\iint_D f(x, y)\, dx\, dy = \iint_{D_1} f(x, y)\, dx\, dy + \iint_{D_2} f(x, y)\, dx\, dy.$$

(v.) $\iint_D 1\, dx\, dy = A(D).$

(vi.) If $m \leq f(x, y) \leq M$ for each $(x, y) \in D$, then

$$m \times A(D) \leq \iint_D f(x, y)\, dx\, dy \leq M \times A(D).$$

Definition 38 *If D is an elementary region, we define the average value of f over D as*

$$\frac{1}{A(D)} \iint_D f(x, y)\, dx\, dy.$$

Definition 39 (Triple integrals) *Let $f : B \mapsto \mathbb{R}$ be a continuous function, where*

$$B = \{(x, y, z) \in \mathbb{R}^3 : \ a \leq x \leq b,\ c \leq y \leq d,\ p \leq z \leq q\}.$$

Consider a partition of $[a,\, b]$ of the form $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$, a partition of $[c,\, d]$ of the form $c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$, and a partition of $[p,\, q]$ of the form $p = z_0 < z_1 < \cdots < z_{\ell-1} < z_\ell = q$.

Choose these partitions such that $x_i - x_{i-1} = (b-a)/m =: \Delta x$ for each $i = 1, \ldots, m$, $y_j - y_{j-1} = (d-c)/n =: \Delta y$ for each $j = 1, \ldots, n$ and $z_k - z_{k-1} = (q-p)/\ell =: \Delta z$ for each $k = 1, \ldots, \ell$ so that each cuboid $B_{i,j.k} = [x_{i-1},\, x_i] \times [y_{j-1},\, y_j] \times [z_{k-1},\, z_k]$ has volume $\Delta V = \Delta x \Delta y \Delta z$. We define the integral of f over B by the following limit

$$\iiint_B f(x, y)\, dV := \lim_{\ell, m, n \to \infty} \sum_{k=0}^{\ell} \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i^*,\, y_j^*,\, z_k^*) \Delta V$$

where x_i^* denotes a point in the interval $[x_{i-1},\, x_i]$, y_j^* denotes a point in the interval $[y_{j-1},\, y_j]$ and z_k^* denotes a point in the interval $[z_{k-1},\, z_k]$.

Definition 40 (Elementary regions in $\mathbb{R}^3$) *Let $E \subset \mathbb{R}^3$. E is a type–1 region of $\mathbb{R}^3$ if it can be expressed in the form*

$$E = \{(x, y, z) : \ (x, y) \in D,\ g_1(x, y) \leq z \leq g_2(x, y)\},$$

where D is the projection of E onto the xy-plane and $g_1, g_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ are continuous functions.

E is a type–2 region of $\mathbb{R}^3$ if it can be expressed in the form

$$E = \{(x, y, z) : (y, z) \in D,\ h_1(y, z) \le x \le h_2(y, z)\},$$

where D is the projection of E onto the yz-plane and $h_1, h_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ are continuous functions.
E is a type–3 region of $\mathbb{R}^3$ if it can be expressed in the form

$$E = \{(x, y, z) : (x, z) \in D,\ f_1(x, z) \le y \le f_2(x, z)\},$$

where D is the projection of E onto the xz-plane and $f_1, f_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ are continuous functions.
E is an elementary region of $\mathbb{R}^3$ if it is a type–1, type–2, type–3 region or a cuboid.

Theorem 14 (Fubini for triple integrals) *If $f : B \mapsto \mathbb{R}$ is continuous, where*

$$B = \{(x, y, z) \in \mathbb{R}^3 : a \le x \le b,\ c \le y \le d,\ p \le z \le q\},$$

then

$$\iiint_B f(x, y, z)\, dV = \int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

Definition 41 *A mapping $T : A \mapsto A$ is one–to–one if for x and $x' \in A$, $T(x) = T(x')$ implies that $x = x'$.*

Remark 1 *Recall that if $f : \mathbb{R} \mapsto \mathbb{R}$ is an integrable function and $g : \mathbb{R} \mapsto \mathbb{R}$ is C^1, then*

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_a^b f(g(u))g'(u)\, du.$$

If g is one–to-one, then it is either nondecreasing or nonincreasing and we have that

$$\int_{g([a,b])} f(x)\, dx = \int_a^b f(g(u))|g'(u)|\, du.$$

where $g([a, b])$ denotes the image of $[a, b]$ under g.

Definition 42 (Jacobian) *A C^1 mapping from $\mathbb{R}^2$ to itself is a map $T(u, v) = (x, y)$ where $x = g(u, v)$ and $y = h(u, v)$ such that $g, h : \mathbb{R}^2 \mapsto \mathbb{R}$ are C^1.*
The Jacobian of the map T is given by

$$J_T = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 15 (Change of variables theorem) *Let E and E^* be elementary regions in $\mathbb{R}^2$ and suppose $T : E^* \mapsto E$ is a one–to–one mapping which is also C^1. Suppose further that $T(E^*) = E$, i.e. E is the image of E^* under the mapping T. For any integrable function $f : E \mapsto \mathbb{R}$, we have*

$$\iint_E f(x, y)\, dx\, dy = \iint_{E^*} f(x(u, v),\ y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

Chapter 5

Vector Calculus

Definition 43 *Let $D \subset \mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ that assigns to each point $(x, y) \in D$ the two dimensional vector $F(x, y)$.*

Example 2 *If $f : \mathbb{R}^2 \mapsto \mathbb{R}$, then $\nabla f(x, y)$ defines a vector field on $\mathbb{R}^2$.*

Definition 44 (Conservative Vector Field) *A vector field $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is called a conservative vector field if there exists an $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $F(x, y) = \nabla f(x, y)$.*

Definition 45 (Line Integral) *Let C be a smooth curve defined by*

$$C = \{(x(t), y(t)) : t \in [a, b]\}.$$

C smooth means that x and y are C^1 functions from $\mathbb{R}$ to $\mathbb{R}$. Consider a partition of $[a, b]$ of the form

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

and divide C into n equal length subarcs, i.e.

$$C_i = \{(x(t), y(t)) : t \in [t_{i-1}, t_i]\}, \quad i \in \{1, \ldots, n\}$$

where $t_i - t_{i-1} = \Delta t$ for each $i \in \{1, \ldots, n\}$. We define the line integral of the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ over C by

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=0}^{n} f\left(x(t_i^*),\, y(t_i^*)\right) \Delta s_i$$

where $t_i^* \in [t_{i-1}, t_i]$ for each $\in \{1, \ldots, n\}$ and Δs_i is the length of the i-th subarc C_i.

Proposition 9 *If C is a smooth curve as in Definition 45 and $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is integrable, then*

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Definition 46 *The line integral of f over C with respect to x is given by*

$$\int_C f(x,y)\ dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt$$

and the line integral of f over C with respect to y is given by

$$\int_C f(x,y)\ dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt.$$

Definition 47 *Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r} : [a,b] \mapsto \mathbb{R}^2$. Then the line integral of F along C is*

$$\int_C F \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

Theorem 16 *Suppose C is a smooth curve given by a vector function $\vec{r} : [a,b] \mapsto \mathbb{R}^2$. If $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a differentiable function, then*

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Definition 48 *We say that a line integral is path independent if for any two smooth curves C_1 and C_2 which begin and end at the same point, we have*

$$\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}.$$

Theorem 17 *Suppose $D \subset \mathbb{R}^2$. $\int_C F \cdot d\vec{r}$ is path independent in D if and only if $\int_C F \cdot d\vec{r} = 0$ for every closed curve C in D (a curve is closed if it starts and ends at the same point).*

Theorem 18 *Suppose F is a vector field that is continuous on an open connected region $D \subset \mathbb{R}^2$. If $\int_C F \cdot d\vec{r}$ is independent of path on D, then F is a conservative vector field on D.*

Theorem 19 *If $F = P\vec{i} + Q\vec{j}$ is a conservative vector field on $D \subset \mathbb{R}^2$, with P, Q C^1 on D, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D.$$

Theorem 20 *Let F be a vector field on an open simply–connected region $D \subset \mathbb{R}^2$. Suppose P, Q are C^1 on D and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D,$$

then F is conservative on D.

Theorem 21 *If $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field on $D \subset \mathbb{R}^2$ with P, Q continuously differentiable on D, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D.$$

Definition 49 *A curve is simple if it does not intersect itself.*

Definition 50 *A region $D \subseteq \mathbb{R}^2$ is simply–connected if every simple closed curve in D encloses only points contained in D ("D has no holes").*

Theorem 22 *Let F be a vector field on an open and simply–connected region $D \subset \mathbb{R}^2$. Suppose P, Q are C^1 on D and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D,$$

then F is conservative on D.

Theorem 23 (Green's Theorem) *Suppose C is a positively oriented (i.e. counterclockwise), piecewise–C^1, simple closed curve in $\mathbb{R}^2$. Denote by D the region of the plane enclosed by C, i.e. $\partial D = C$. $E \subset \mathbb{R}^2$ is an open set such that $D \subset E$ and $P, Q : \mathbb{R}^2 \mapsto \mathbb{R}$ are C^1 on E. Then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Definition 51 (Curl) *Let $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a vector field. The curl of the vector field F is given by*

$$curl(F) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k},$$

where $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$.

Remark 2 *If ∇ denotes the gradient operator and $\times$ the cross product, then we have the following "formal" expression for the curl:*

$$curl(F) = \nabla \times F.$$

Theorem 24 *If $f : \mathbb{R}^3 \mapsto \mathbb{R}$ has continuous second order partials, then*

$$curl(\nabla f) = 0,$$

i.e. conservative vector fields have zero curl.

Remark 3 *The contrapositive form of the previous theorem is very useful: a vector field with nonzero curl is not conservative.*

Theorem 25 *Suppose $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a vector field whose components are C^1 on all of $\mathbb{R}^3$. If $curl(F) = 0$, then F is conservative.*

Definition 52 (Divergence) *Let $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a vector field. The divergence of the vector field F is given by*

$$div(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

where $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$.

Remark 4 *We have the following "formal" expression for the divergence:*

$$div(F) = \nabla \cdot F$$

Theorem 26 *If $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ such that P, Q and R are C^2, then*

$$div(curl(F)) = 0.$$

Remark 5 *We can rewrite Green's Theorem using the divergence and curl as follows:*
If $F = P\vec{i} + Q\vec{j} + 0\,\vec{k}$, then

$$\int_C P\,dx + Q\,dy = \int_C F \cdot \vec{n}\,ds = \iint_D div(F)\,dA = \iint_D curl(F) \cdot \vec{k}\,dA,$$

where $\vec{n}$ is the unit normal to the curve C.

Bonus Material (higher dimensions):

Suppose $x, y, z : \mathbb{R}^2 \mapsto \mathbb{R}$ and thus let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ describe the 3–dimensional surface S. The integral of $f : \mathbb{R}^3 \mapsto \mathbb{R}$ over the surface S parameterised by $\vec{r}$ is given by

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v)) \left| \frac{\partial}{\partial u}\vec{r}(u,v) \times \frac{\partial}{\partial v}\vec{r}(u,v) \right| dA,$$

where D is the projection of S onto $\mathbb{R}^2$.

The integral of the vector field $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ over the surface S (defined as above) is

$$\iint_S F \cdot dS := \iint_S F \cdot \vec{n}\,dS = \iint_D F \cdot \left(\frac{\partial}{\partial u}\vec{r}(u,v) \times \frac{\partial}{\partial v}\vec{r}(u,v) \right) dA$$

where $\vec{n}$ is the unit normal vector to S.

Theorem 27 (Stoke's Theorem) *Let S be an oriented piecewise smooth surface that is bounded by a simple piecewise smooth closed boundary curve C with positive orientation. Let F be a C^1 vector field on an open region in $\mathbb{R}^3$ which contains S. Then*

$$\iint_C F \cdot d\vec{r} = \iint_S curl(F) \cdot dS.$$